

Wednesday, October 29, Lecture 25: Intro to Fourier series,
Heat eqn

• New Subjects!

Imagine a temperature distribution $u(r, \theta)$ on a disc, which satisfies heat eqn:

$$\frac{\partial}{\partial t} u = \Delta u$$

Let $t \rightarrow \infty$, so steady state is Laplace's equation:
 $0 = \Delta u.$

In polar coordinates,

$$0 = \frac{\partial^2}{\partial r^2} u + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

$$\Rightarrow r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} = - \frac{\partial^2 u}{\partial \theta^2}$$

Guess: $u(r, \theta) = F(r)T(\theta)$ (separation of variables ansatz)

so

$$\frac{r^2 F''(r) + r F'(r)}{F(r)} = - \frac{T''(\theta)}{T(\theta)}$$

But these are independent variables, so they must be equal to constants, say

$$k^2 = - \frac{T''(\theta)}{T(\theta)}, \text{ or}$$

$T''(\theta) + k^2 T(\theta) = 0$. Let's focus on this.

Def: Periodic

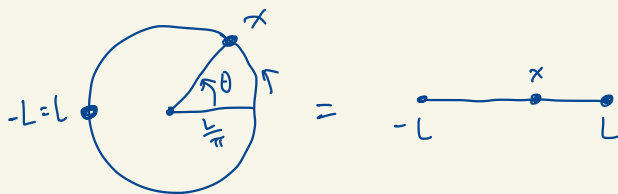
A function $f: \mathbb{R} \rightarrow \mathbb{C}$ is called periodic with period $2L$ if

$$f(x+2L) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

Smallest such $2L$ = fundamental period

3 equivalent perspectives:

- $f: \mathbb{R} \rightarrow \mathbb{C}$ is periodic with period $2L$
- $f: [-L, L] \rightarrow \mathbb{C}$ satisfies periodic boundary conditions, $f(-L) = f(L)$
- $f: S^1 \rightarrow \mathbb{C}$ is a function on the circle of radius $\frac{L}{\pi}$, so $\theta \cdot \frac{L}{\pi} = x$ "position on



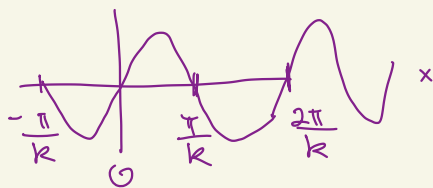
To be a function, this requires "periodic boundary conditions" $f(-L) = f(L)$

Ex: $\sin x$ has period 2π .

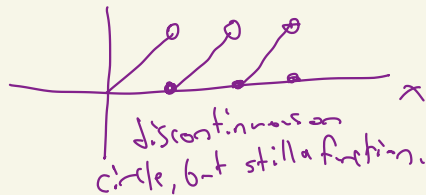
$\sin kx$ has period $\frac{2\pi}{k}$

check:

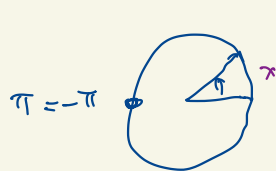
$$\sin 0 = \sin k\left(\frac{2\pi}{k}\right)$$



"Sawtooth"



Back to temperature, say on the unit circle, so $x = \theta$.



$$T: [-\pi, \pi] \rightarrow \mathbb{R}$$

$$T(-\pi) = T(\pi),$$

$$T(-\pi) = T'(\pi)$$

"periodic PBC boundary conditions" ensure T is defined on circle.

• So $T(x)$ is 2π -periodic.

$$T'' + k^2 T = 0$$

or

$$\left(\frac{d}{dx}\right)^2 T + k^2 T = 0$$

• First, we find a couple basic solutions:

$$\cdot (\sin kx)'' + k^2 (\sin kx) = -k^2 \sin kx + k^2 \sin kx = 0 \quad \checkmark$$

Same for $\cos kx$.

• One more

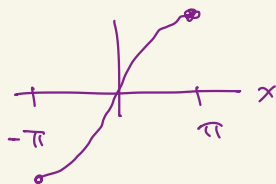
$$(e^{ikx})'' + k^2 e^{ikx} = i^2 k^2 e^{ikx} + k^2 e^{ikx} = 0.$$

• PBC forces k to be an integer.

Otherwise, T not defined on this circle (i.e. not 2π -periodic).

Ex:

$$\sin \frac{1}{2}x$$



$$\sin\left(\frac{1}{2}(-\pi)\right) = -1$$

$$\sin\left(\frac{1}{2}\pi\right) = 1.$$

So not a function on the circle

- HW: Function spaces are vector spaces, and derivatives can be thought of as linear operators, so this is actually an eigenvalue problem:

$$\begin{array}{ccc} \left(\frac{d}{dx}\right)^2 T = -k^2 T \\ \uparrow \quad \quad \uparrow \\ \text{linear operator} \quad \text{eigenvalue} \end{array}$$

Remember from linear algebra:

Diagonalize linear operator $\Leftrightarrow$ find a basis of eigenvectors $\{T_k\}$

In this basis, we have

$$DT_k = -k^2 T_k$$

- Remember, diagonal bases make our lives easier!
- we have eigenvectors: $\cos kx$, $\sin kx$, e^{ikx} , $k \in \mathbb{Z}$!
- So, to characterize solutions T to (*), we expand T in a basis of eigenvectors:
- (Exponential) Fourier Series

$$T(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$
- (Trigonometric) Fourier Series:

$$T(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

The two are related by

$$e^{ikx} = \cos kx + i \sin kx \quad k \geq 0$$

$$\begin{aligned} e^{-ikx} &= \cos(-kx) + i \sin(-kx) \\ &= \cos kx - i \sin kx \end{aligned} \quad \begin{array}{l} \cos \text{ is even,} \\ \sin \text{ is odd} \end{array}$$

so $k \geq 0$,

$$\begin{aligned} c_k e^{ikx} + c_{-k} e^{-ikx} &= c_k (\cos kx + i \sin kx) + c_{-k} (\cos kx - i \sin kx) \\ &= (c_k + c_{-k}) \cos kx + i(c_k - c_{-k}) \sin kx \end{aligned}$$

$$\Rightarrow a_k = c_k + c_{-k} \quad (\text{so } a_0 = 2c_0, \text{ hence } \frac{a_0}{2})$$

$$b_k = \frac{c_k - c_{-k}}{i}$$

- Shift in perspective: instead of studying T directly as a function, write it in this nice basis and study its Fourier coefficients c_k .

- Many questions ...
 - Can we really just take infinite sums and pretend it's linear algebra?
 - Series, so need notion of convergence.
 - This dictates which functions we can do this trick for!
 - How do we find these series?
 - How to better understand this representation?
 - What can we do with it?

Where we're going:

- Hilbert spaces = function spaces with inner product.
 - "Linear algebra + convergence"
- Back to Fourier analysis